TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 252, August 1979

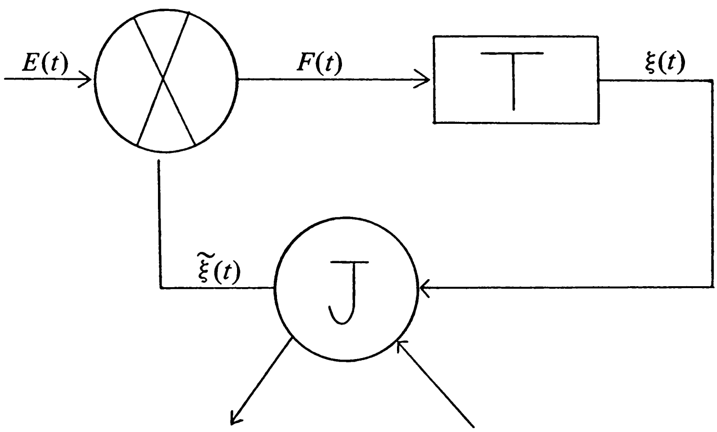
THEOREMS OF FUBINI TYPE FOR ITERATED STOCHASTIC INTEGRALS

BY

MARC A. BERGER[1] AND VICTOR J. MIZEL[2]

ABSTRACT. An extension of the Itô calculus which treats iterated Itô integration, as applied to a class of two-parameter processes, is introduced. This theory includes the integration of certain *anticipative* integrands and introduces a notion of stochastic differential for such integrands. Among the key results is a version of Fubini's theorem for iterated stochastic integrals, in which a "correction" term appears. Applications to stochastic integral equations and to the Itô calculus are given, and the relation of the present development to recent work of Ogawa is described.

1. Introduction. Shown in the figure below is a typical feedback diagram.



The box T signifies a transfer from the input F to the output ξ. For example,

$$\xi(t) = \int_0^t \sigma(t-\tau)F(\tau)\,d\tau, \qquad t \geqslant 0. \tag{1.1}$$

Junction J is a step-up or step-down point. Here either some fraction of ξ is

Received by the editors February 6, 1978.

AMS (MOS) subject classifications (1970). Primary 60H20, 45D05.

Key words and phrases. Brownian motion, stochastic integral, Volterra equation.

[1]Sponsored by the Fannie and John Hertz Foundation Fellowship Fund, the United States Army under Contract No. DAAG29-75-C-0024 and by the National Science Foundation under Grant No. MCS75-17385 A01.

[2]Partially supported by National Science Foundation Grant MCS71-02776 A05, by a sabbatical grant from Carnegie-Mellon University, and by the Lady Davis Fellowship Fund.

0002-9947/79/0000-0362/$07.50

diverted for external consumption, or else ξ is scaled up. Thus the remainder in the loop is

$$\tilde{\xi} = \alpha\xi. \tag{1.2}$$

If the process uses this remainder $\tilde{\xi}$ to drive itself, along with an external driving force E, then

$$F = E + \tilde{\xi}. \tag{1.3}$$

Combining (1.1), (1.2), (1.3) it follows that the equation governing the system is

$$\xi(t) - \int_0^t \sigma(t-\tau)\alpha(\tau)\xi(\tau)\,d\tau = \int_0^t \sigma(t-\tau)E(\tau)\,d\tau, \qquad t \geqslant 0. \tag{1.4}$$

Suppose, however, that α is in the form of a noise, $\alpha = \alpha_1 + \alpha_2 z$, where z is a white noise. Then (1.4) becomes

$$\xi(t) - \int_0^t \sigma(t-\tau)\alpha_1(\tau)\xi(\tau)\,d\tau - \int_0^t \sigma(t-\tau)\alpha_2(\tau)\xi(\tau)\,d\beta(\tau)$$
$$= \int_0^t \sigma(t-\tau)E(\tau)\,d\tau, \qquad t \geqslant 0, \tag{1.5}$$

where β is a Brownian motion

$$\beta(t) = \int_0^t z(\tau)\,d\tau, \qquad t \geqslant 0.$$

This equation is an example of a stochastic integral equation. The existence theory of such equations will be discussed in §5. The main difficulty in solving such equations lies in the impossibility of representing the iterates of the operator

$$Tf(t) = \int_0^t \sigma(t-\tau)\alpha_2(\tau)f(\tau)\,d\beta(\tau), \qquad t \geqslant 0, \tag{1.6}$$

in a form similar to (1.6). This difficulty arises precisely because the integrand in a stochastic integral must be nonanticipating. Thus there is no meaning given a priori to an integral like

$$\int_0^t \left[\int_\tau^t \sigma(t-\tau_1)\sigma(\tau_1-\tau)\alpha_2(\tau_1)\,d\beta(\tau_1) \right] \alpha_2(\tau)f(\tau)\,d\beta(\tau). \tag{1.7}$$

Itô **[8]** has defined an integral

$$I(t) = \int_0^t \int_0^t g(\tau_1, \tau_2)\,d\beta(\tau_1)\,d\beta(\tau_2)$$

where $g \in L^2([0, t] \times [0, t])$. His definition there is

$$I(t) = \int_0^t \left\{ \int_0^{\tau_2} [g(\tau_1, \tau_2) + g(\tau_2, \tau_1)]\,d\beta(\tau_1) \right\} d\beta(\tau_2), \qquad t \geqslant 0. \tag{1.8}$$

This integral behaves in many ways like a single stochastic integral, but *not* like two iterated integrals. For example, one can show that (1.8) implies

$$\int_0^t \int_0^t \varphi(\tau_1)\psi(\tau_2)\, d\beta(\tau_1)\, d\beta(\tau_2)$$
$$= \left[\int_0^t \varphi(\tau)\, d\beta(\tau)\right]\left[\int_0^t \psi(\tau)\, d\beta(\tau)\right] - \int_0^t \varphi(\tau)\psi(\tau)\, d\tau, \qquad t \geq 0, \quad (1.9)$$

for φ, $\psi \in L^2([0, t])$. Thus, although according to (1.8) the natural definition for integrals like (1.7) should be

$$\int_0^t \int_\tau^t g(\tau, \tau_1, t)\, d\beta(\tau_1)\, d\beta(\tau) = \int_0^t \int_0^\tau g(\tau_1, \tau, t)\, d\beta(\tau_1)\, d\beta(\tau), \qquad t \geq 0,$$

this interpretation has the disadvantage that it utilizes what is in actuality a two-dimensional integral, rather than an iterated one-dimensional integral.

For this reason we develop here a different extension of the stochastic integral, which allows one to solve equations like (1.5) by iterating operators such as that appearing in (1.6). This extension is, roughly speaking, the unique extension which allows integrals to be iterated one variable at a time, in the usual fashion. Thus, for example, a formula like (1.9) will be replaced by

$$\int_0^t \int_0^t \varphi(\tau_1)\psi(\tau_2)\, d\beta(\tau_1)\, d\beta(\tau_2) = \left[\int_0^t \varphi(\tau)\, d\beta(\tau)\right]\left[\int_0^t \psi(\tau)\, d\beta(\tau)\right], \qquad t \geq 0.$$

The distinction between our integral and that of Itô, defined by (1.8), will be clarified through the Correction Formula (Theorem 3.A). The ease of calculating with our integral readily enables one to uncover a number of important properties of the Itô stochastic calculus. For example, in Theorem 4.B we provide a differentiation rule for processes of the form

$$\xi(t) = F(t, \beta(t)), \qquad t \geq 0,$$

where

$$F(t, x) = \int_0^t \varphi(\tau, t, x - \beta(\tau))\, d\beta(\tau), \qquad t \geq 0, x \in \mathbf{R}.$$

For a different approach to the Correction Formula the reader is referred to Meyer [**14**, pp. 321–326]. For reference to other types of random integral equations, we refer the reader to the comprehensive works by Bharucha-Reid [**4**] and Tsokos and Padgett [**17**].

2. Adapted stochastic integral. Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space, and $\{\beta(t): t \geq 0\}$ a Brownian motion on it. For $0 \leq t_1 \leq t_2$ let $\mathfrak{F}(t_1, t_2)$ denote the sub-sigma-algebra of $\mathfrak{F}$ generated by $\{\beta(\tau) - \beta(t_1): t_1 \leq \tau \leq t_2\}$. A two-parameter stochastic process $\{f(t_1, t_2): 0 < t_1 < t_2\}$ is said to be L^2_+-adapted (with respect to β) if

(i) $f(\cdot, t_2)$ is separable and measurable on $[0, t_2]$, $t_2 > 0$,

(ii) $f(t_1, t_2)$ is $\mathfrak{F}(t_1, t_2)$-measurable, $0 < t_1 < t_2$,

(iii) $f(t_1, t_2) \in L^2(\Omega)$, $0 < t_1 < t_2$,

(iv) $\int_{t_1}^{t_2} \mathbf{E}|f(\tau, t_2)|^2 \, d\tau < \infty$, $t_2 \geq 0$.

If conditions (i) and (iv) are replaced by

(i)′ $f(t_1, \cdot)$ is separable and measurable on $[t_1, \infty)$, $t_1 > 0$,

(iv)′ $\int_{t_1}^{t_2} \mathbf{E}|f(t_1, \tau)|^2 \, d\tau < \infty$, $0 < t_1 < t_2$,

then f is said to be L^2_--adapted (with respect to β).

Itô [7] has defined the integral $\int_{t_1}^{t_2} f(t_1, \tau) \, d\beta(\tau)$ for L^2_--adapted processes f, and its properties can be found in any text on stochastic integration. (See, for example, Arnold [**1**, pp. 64–88], Friedman [**5**, pp. 59–72], Gihman and Skorohod [**6**, pp. 11–27], McKean [**12**, pp. 24–29], McShane [**13**, pp. 102–152], Skorohod [**16**, pp. 15–29].) We address ourselves to the problem of defining a new stochastic integral of the form $\int_{t_1}^{t_2} f(\tau, t_2) \, d\beta(\tau)$ for L^2_+-adapted processes f.

To begin with we establish the following result characterizing L^2_+-adapted processes.

THEOREM 2.A. *Let $T_n(t_1, t_2)$ denote the region*

$$\{(\tau_1, \ldots, \tau_n): t_1 \leq \tau_1 \leq \cdots \leq \tau_n \leq t_2\}, \qquad 0 \leq t_1 \leq t_2.$$

For any L^2_+-adapted process f there exists a unique sequence

$$\{\varphi_n(t_1, t_2) \in L^2(T_n(t_1, t_2)): 0 \leq t_1 \leq t_2, n = 1, 2, \ldots\}$$

such that, for $0 \leq t_1 \leq t_2$, $f(t_1, t_2)$ has the $L^2(\Omega)$ orthogonal expansion

$$\mathbf{E}f(t_1, t_2) + \sum_{n=1}^{\infty} \int_{T_n(t_1, t_2)} \varphi_n(t_1, t_2; \tau_1, \ldots, \tau_n) \, d\beta(\tau_1) \cdots d\beta(\tau_n). \tag{2.1}$$

As a consequence, for $0 \leq t_1 \leq t_2$,

$$f(t_1, t_2) = \mathbf{E}f(t_1, t_2) + \int_{t_1}^{t_2} \psi(t_1, \tau, t_2) \, d\beta(\tau) \tag{2.2}$$

where $\psi(t_1, \tau, t_2)$ is $\mathfrak{F}(t_1, \tau)$-measurable, a.e. $\tau \in [t_1, t_2]$, and

$$\mathbf{E}\int_0^{\tau} |\psi(t, \tau, t_2)|^2 \, dt < \infty, \qquad t_2 \geq 0, a.e. \ \tau \in [t_1, t_2], \tag{2.3}$$

$$\mathbf{E}\int_{t_1}^{t_2} |\psi(t_1, \tau, t_2)|^2 \, d\tau < \infty, \qquad 0 \leq t_1 \leq t_2. \tag{2.4}$$

PROOF. By considering the Brownian motion

$$\beta_*(t) = \beta(t_1 + t) - \beta(t_1), \qquad 0 \leq t \leq t_2 - t_1,$$

the expansion (2.1) becomes a form of the homogeneous chaos, and follows directly from Theorem 4.2 and Theorem 5.1 of Itô [8]. The uniqueness follows from Theorem 4.3 there. The fact that f is L_+^2-adapted implies that for $t_2 > 0$

$$\mathbf{E}\int_0^{t_2}\int_0^{\tau}|\psi(t,\tau,t_2)|^2\,dt\,d\tau = \mathbf{E}\int_0^{t_2}\int_t^{t_2}|\psi(t,\tau,t_2)|^2\,d\tau\,dt$$

$$\leqslant \mathbf{E}\int_0^{t_2}|f(t,t_2)|^2\,dt < \infty,$$

from which (2.3) follows. Similarly, for (2.4),

$$\mathbf{E}\int_{t_1}^{t_2}|\psi(t_1,\tau,t_2)|^2\,d\tau \leqslant \mathbf{E}|f(t_1,t_2)|^2 < \infty. \quad \square$$

It will be necessary in what follows to restrict our attention to L_+^2-adapted processes f for which a stochastic differential of the type $\partial_{t_1} f(t_1, t_2)$ exists. That is, using the notation of Theorem 2.A, we will require that

(i) $\partial\,\mathbf{E}f(t_1,t_2)/\partial t_1$ exists, and the left strong L^2-derivative $\partial\varphi_n(t_1,t_2)/\partial t_1$ exists as an element in $L^2(T_n(t_1,t_2))$,[3] $0 \leqslant t_1 \leqslant t_2$, $n = 1, 2, \ldots$.

(ii) The series

$$\frac{\partial}{\partial t_1}\mathbf{E}f(t_1,t_2) + \sum_{n=1}^{\infty}\int_{T_n(t_1,t_2)}\frac{\partial}{\partial t_1}\varphi_n(t_1,t_2;\tau_1,\ldots,\tau_n)\,d\beta(\tau_1)\cdots d\beta(\tau_n)$$

converges in $L^2(\Omega)$, for $0 \leqslant t_1 \leqslant t_2$, to an L_+^2-adapted process $f^0(t_1,t_2)$.

(ii) $\varphi_1(t_1,t_2;t_1)$ is continuous for $0 \leqslant t_1 \leqslant t_2$, and $\varphi_n(t_1,t_2;t_1,\cdot,\ldots,\cdot)$ exists (as a trace) in $L^2(T_{n-1}(t_1,t_2))$, $0 \leqslant t_1 \leqslant t_2$, $n = 2, 3, \ldots$.

(iv) The series

$$\varphi_1(t_1,t_2;t_1) + \sum_{n=2}^{\infty}\int_{T_{n-1}(t_1,t_2)}\varphi_n(t_1,t_2;t_1,\tau_1,\ldots,\tau_{n-1})\,d\beta(\tau_1)\cdots d\beta(\tau_{n-1})$$

converges in $L^2(\Omega)$, for $0 \leqslant t_1 \leqslant t_2$, to an L_+^2-adapted process $f'(t_1,t_2)$.[4]

Such processes f are said to be $L_+^{2,1}$-adapted (with respect to β). The process f' is called the *diffusional part of f*. In many ways it behaves like a derivative. For example, if f is of the form

$$f(t_1,t_2) = F(\beta(t_2) - \beta(t_1)), \qquad 0 \leqslant t_1 \leqslant t_2,$$

where $F \in C^1(\mathbf{R})$, then

$$f'(t_1,t_2) = F'(\beta(t_2) - \beta(t_1)), \qquad 0 \leqslant t_1 \leqslant t_2.$$

In fact, if f' is $L_+^{2,1}$-adapted, then $\partial_{t_1} f(t_1,t_2)$ formally exists and is given by

[3] See Lions and Magenes [11, pp. 191–192]. Note that (iii) follows from this statement.

[4] In fact, the above conditions hold if $\varphi_n(\tau_1, t_2; \tau_2, \ldots, \tau_{n+1})$ belongs to the Sobolev space $W^{1,2}(T_{n+1}(0,t_2))$, $n = 1, 2, \ldots$, and the norms satisfy $\sum_{n=1}^{\infty}\|\varphi_n(\cdot,t_2)\|^2_{W^{1,2}(T_{n+1}(0,t_2))} < \infty$, $t_2 > 0$. See, for example, Kufner, John and Fucik [10].

$$\partial_{t_1} f(t_1, t_2) = \left[f^0(t_1, t_2) + f^\frown(t_1, t_2) \right] dt_1 - f^\frown(t_1, t_2)\, d\beta(t_1), \qquad 0 < t_1 < t_2. \tag{2.5}$$

Hence $f^\frown$ is simply the negative of the diffusion term in $\partial_{t_1} f(t_1, t_2)$.

We now make the following definition. Suppose f is an $L_+^{2,1}$-adapted process of the monomial form

$$f(t_1, t_2) = \int_{T_n(t_1,t_2)} \varphi(t_1, t_2; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$$

where $\varphi(t_1, t_2) \in L^2(T_n(t_1, t_2))$, $0 < t_1 < t_2$. Then $\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)$ is defined to be the process[5]

$$I(t_1, t_2) = \int_{T_{n+1}(t_1,t_2)} \varphi(\tau_1, t_2; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1})$$
$$+ \int_{t_1}^{t_2} f^\frown(\tau, t_2)\, d\tau, \qquad 0 < t_1 < t_2. \tag{2.6}$$

The first term on the right of (2.6) exists since f is L_+^2-adapted. The motivation for this definition is that it follows now from (2.5) formally that

$$\partial_{t_1} I(t_1, t_2) = -f(t_1, t_2)\, d\beta(t_1), \qquad 0 < t_1 < t_2.$$

We note that

$$\mathbf{E}\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau) = \mathbf{E}\int_{t_1}^{t_2} f^\frown(\tau, t_2)\, d\tau, \qquad 0 < t_1 < t_2. \tag{2.7}$$

and that this is zero if $n > 1$. Furthermore if $\tilde{f}$ is an $L_+^{2,1}$-adapted process of the monomial form

$$\tilde{f}(t_1, t_2) = \int_{T_m(t_1,t_2)} \tilde{\varphi}(t_1, t_2; \tau_1, \ldots, \tau_m)\, d\beta(\tau_1) \cdots d\beta(\tau_m)$$

where $m < n$ and $\tilde{\varphi}(t_1, t_2) \in L^2(T_m(t_1, t_2))$, $0 < t_1 < t_2$, then

---

[5]By repeated application of a Fubini theorem for iterated integrals of the form $dt\, d\beta(t)$ to the second term on the right of (2.6), one obtains

$$I(t_1, t_2) = \int_{T_{n+1}(t_1,t_2)} \varphi(\tau_1, t_2; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1})$$
$$+ \int_{T_{n-1}(t_1,t_2)} \left[\int_{t_1}^{\tau_1} \varphi(\tau, t_2; \tau, \tau_1, \ldots, \tau_{n-1})\, d\tau \right] d\beta(\tau_1) \cdots d\beta(\tau_{n-1}), \quad 0 < t_1 < t_2.$$

$$\mathbf{E}\left[\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)\right]\left[\int_{t_1}^{t_2} \tilde{f}\,(\tau, t_2)\, d\beta(\tau)\right]$$

$$= \begin{cases} \mathbf{E}\left\{\int_{t_1}^{t_2} f(\tau, t_2)\tilde{f}\,(\tau, t_2)\, d\tau + \left[\int_{t_1}^{t_2} f'(\tau, t_2)\, d\tau\right]\left[\int_{t_1}^{t_2} \tilde{f}'(\tau, t_2)\, d\tau\right]\right\}, & \\ \qquad\qquad m = n, & \\ \int_{T_n(t_1,t_2)} \varphi(\tau_1, t_2; \tau_1, \ldots, \tau_n)\tilde{\varphi}(\tau_2, t_2; \tau_3, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n, & \\ \qquad\qquad m = n - 2, & \\ 0, \quad \text{otherwise.} & \end{cases} \tag{2.8}$$

Here we have used the orthogonality of monomials of order m and n, where $m \neq n$. In particular,

$$\mathbf{E}\left|\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)\right|^2$$
$$= \mathbf{E}\left[\int_{t_1}^{t_2} |f(\tau, t_2)|^2\, d\tau + \left|\int_{t_1}^{t_2} f'(\tau, t_2)\, d\tau\right|^2\right], \qquad 0 < t_1 < t_2. \tag{2.9}$$

Using the above definition and Theorem 2.A it is now desired to extend this stochastic integral to general $L_+^{2,1}$-adapted processes f. To this end the following result is presented.

THEOREM 2.B. *There is a unique extension of the integral* $\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)$ *to all* $L_+^{2,1}$*-adapted processes* f*, satisfying the following continuity condition*:

Whenever $\{f_k: k = 1, 2, \ldots\}$ *is a sequence of* $L_+^{2,1}$*-adapted processes for which*

$$\lim_{k\to\infty} \mathbf{E}\left[|f_k(t_1, t_2)|^2 + |f_k^0(t_1, t_2)|^2 + |f_k'(t_1, t_2)|^2\right] = 0,$$

$$\int_0^{t_2}\left\{\sup_{k=1,2,\ldots} \mathbf{E}\left[|f_k^0(\tau, t_2)|^2 + |f_k'(\tau, t_2)|^2\right]\right\} d\tau < \infty,$$

then

$$\lim_{k\to\infty} \mathbf{E}\left|\int_{t_1}^{t_2} f_k(\tau, t_2)\, d\beta(\tau)\right|^2 = 0,$$

for $0 < t_1 < t_2$.

PROOF. For the existence of the extension it is necessary to establish the $L^2(\Omega)$ convergence of the series

$$\sum_{n=1}^{\infty} \int_{t_1}^{t_2} f_n(\tau, t_2)\, d\beta(\tau), \qquad 0 < t_1 < t_2, \tag{2.10}$$

for each f which has the $L^2(\Omega)$ expansion

$$f(t_1, t_2) = \mathbf{E}f(t_1, t_2) + \sum_{n=1}^{\infty} f_n(t_1, t_2), \qquad 0 < t_1 < t_2,$$

with f_n $L_+^{2,1}$-adapted and of the monomial form

$$f_n(t_1, t_2) = \int_{T_n(t_1,t_2)} \varphi_n(t_1, t_2; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \qquad 0 < t_1 < t_2,$$

for $n = 1, 2, \ldots$. Here, as before,

$$\varphi_n(t_1, t_2) \in L^2(T_n(t_1, t_2)), \qquad 0 < t_1 < t_2, n = 1, 2, \ldots.$$

Consider first the series

$$\sum_{n=1}^{\infty} \int_{t_1}^{t_2} f_n^{\cdot}(\tau, t_2)\, d\tau. \tag{2.11}$$

By assumption (iv) in the definition of $L_+^{2,1}$-adapted, the series $\sum_{n=1}^{N} f_n^{\cdot}(\tau, t_2)$ tends to $f^{\cdot}(\tau, t_2)$ in $L^2(\Omega)$ for large N, at each point $\tau \in [t_1, t_2]$. Furthermore, again using the orthogonality of monomials of different order,

$$\sum_{n=1}^{\infty} \mathbf{E}|f_n^{\cdot}(t_1, t_2)|^2 = \mathbf{E}|f^{\cdot}(\tau, t_2)|^2, \qquad \tau \in [t_1, t_2]. \tag{2.12}$$

Since $f^{\cdot}$ is L_+^2-adapted,

$$\int_{t_1}^{t_2} \mathbf{E}|f^{\cdot}(\tau, t_2)|^2\, d\tau < \infty, \qquad 0 < t_1 < t_2.$$

Thus, by the Monotone Convergence Theorem, the series

$$\sum_{n=1}^{N} \int_{t_1}^{t_2} \mathbf{E}|f_n^{\cdot}(t_1, t_2)|^2\, d\tau$$

tends to $\int_{t_1}^{t_2} \mathbf{E}|f^{\cdot}(\tau, t_2)|^2\, d\tau$ for large N. Furthermore, by the Cauchy-Schwarz inequality

$$\mathbf{E}\left|\int_{t_1}^{t_2}\left[f^{\cdot}(\tau, t_2) - \sum_{n=1}^{N} f_n^{\cdot}(\tau, t_2)\right] d\tau\right|^2$$

$$< (t_2 - t_1)\int_{t_1}^{t_2} \mathbf{E}\left[|f^{\cdot}(\tau, t_2)|^2 - \sum_{n=1}^{N} |f_n^{\cdot}(\tau, t_2)|^2\right] d\tau, \qquad 0 < t_1 < t_2,$$

and thus by Parseval's Formula the series (2.11) converges to $\int_{t_1}^{t_2} f(\tau, t_2)\, d\tau$ in $L^2(\Omega)$.

Consider next the series

$$\sum_{n=1}^{\infty} g_n(t_1, t_2) \tag{2.13}$$

where

$$g_n(t_1, t_2) = \int_{T_{n+1}(t_1,t_2)} \varphi_n(\tau_1, t_2; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1}),$$

$$0 < t_1 < t_2.$$

By writing

$$\varphi_n(\tau_1, t_2) = \varphi_n(t_1, t_2) + \int_{t_1}^{\tau_1} \frac{\partial}{\partial \tau} \varphi_n(\tau, t_2)\, d\tau$$

and making use of a Fubini theorem for iterated integrals of the form $dt\, d\beta(t)$, one obtains

$$g_n(t_1, t_2) = \int_{T_n(t_1,t_2)} [\beta(\tau_1) - \beta(t_1)] \varphi_n(t_1, t_2; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$$

$$+ \int_{t_1}^{t_2} \left\{ \int_{T_n(\tau,t_2)} [\beta(\tau_1) - \beta(\tau)] \frac{\partial}{\partial \tau} \right.$$

$$\left. \cdot\, \varphi_n(\tau, t_2; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n) \right\} d\tau,$$

$$0 < t_1 < t_2,$$

and thus by the Cauchy-Schwarz inequality

$$\mathbf{E}|g_n(t_1, t_2)|^2 < 2(t_2 - t_1)\mathbf{E}\left[|f_n(t_1, t_2)|^2 + (t_2 - t_1) \int_{t_1}^{t_2} |f_n^0(\tau, t_2)|^2\, d\tau \right],$$

$$0 < t_1 < t_2. \tag{2.14}$$

Since f^0 is L_+^2-adapted the Monotone Convergence Theorem can be used as before to show that

$$\sum_{n=1}^{N} \int_{t_1}^{t_2} \mathbf{E}|f_n^0(\tau, t_2)|^2\, d\tau$$

tends to $\int_{t_1}^{t_2} \mathbf{E}|f^0(\tau, t_2)|^2\, d\tau$ for large N, $0 < t_1 < t_2$. Hence the series converges in $L^2(\Omega)$. And now using (2.6) it follows that the series (2.10) also converges.

Concerning the continuity condition, the estimate

$$\mathbf{E}\left|\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)\right|^2 \leqslant 2\mathbf{E}\left|\int_{t_1}^{t_2} f^{\frown}(\tau, t_2)\, d\tau\right|^2 + 4(t_2 - t_1)\mathbf{E}|f(t_1, t_2)|^2$$

$$+ 4(t_2 - t_1)^2\mathbf{E}\left|\int_{t_1}^{t_2} f^0(\tau, t_2)\, d\tau\right|^2, \qquad 0 \leqslant t_1 \leqslant t_2,$$

which follows from (2.12) and (2.14), and the theorem on dominated convergence, show that the integral we constructed satisfies this condition. □

A closer analysis of the above proof reveals that for the existence of the integral in Theorem 2.B it suffices that $f^{\frown}$ exist, and that

$$\sum_{n=1}^{\infty} \|\varphi_n(\cdot, t_2)\|^2_{L^2(T_{n+1}(0,t_2))} < \infty, \qquad t_2 > 0.$$

Some of the important properties of this integral are summarized in the following result.

THEOREM 2.C. *Let f_1, f_2 be $L^{2,1}_+$-adapted, and set*

$$I_k(t_1, t_2) = \int_{t_1}^{t_2} f_k(\tau, t_2)\, d\beta(\tau), \qquad 0 \leqslant t_1 \leqslant t_2, k = 1, 2.$$

Let $a,b \in \mathbf{R}$. Then

(a) (*Linearity*) $\int_{t_1}^{t_2}[af_1(\tau, t_2) + bf_2(\tau, t_2)]\, d\beta(\tau) = aI_1(t_1, t_2) + bI_2(t_1, t_2)$, $0 \leqslant t_1 \leqslant t_2$.

(b) (*Smoothness*) *I_1 is $L^{2,1}_+$-adapted, and $I_1^0 = -f_1^{\frown}$, $I_1^{\frown} = f_1$.*

(c) $\mathbf{E}I_1(t_1, t_2) = \mathbf{E}\int_{t_1}^{t_2} f_1^{\frown}(\tau, t_2)\, d\tau$, $0 \leqslant t_1 \leqslant t_2$.

(d)

$$\mathbf{E}I_1(t_1, t_2)I_2(t_1, t_2)$$

$$= \mathbf{E}\left\{\int_{t_1}^{t_2} f_1(\tau, t_2) f_2(\tau, t_2)\, d\tau + \left[\int_{t_1}^{t_2} f_1^{\frown}(\tau, t_2)\, d\tau\right]\left[\int_{t_1}^{t_2} f_2^{\frown}(\tau, t_2)\, d\tau\right]\right\}$$

$$+ \mathbf{E}\int_{t_1}^{t_2}\left[f_1^{\frown}(\tau, t_2) g_2(\tau, t_2) + f_2^{\frown}(\tau, t_2) g_1(\tau, t_2)\right] d\tau$$

$$= \mathbf{E}\int_{t_1}^{t_2} f_1(\tau, t_2) f_2(\tau, t_2)\, d\tau$$

$$+ \mathbf{E}\int_{t_1}^{t_2}\left[f_1^{\frown}(\tau, t_2) I_2(\tau, t_2) + f_2^{\frown}(\tau, t_2) I_1(\tau, t_2)\right] d\tau, \quad 0 \leqslant t_1 \leqslant t_2,$$

where

$$g_k(\tau, t_2) = \int_\tau^{t_2} \mathbf{E} f_k(\tau_1, t_2)\, d\beta(\tau_1) + \sum_{n=1}^{\infty} \int_{T_{n+1}(\tau, t_2)} \varphi_{k,n}(\tau_1, t_2; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1}),$$
$$0 < \tau < t_2,\ k = 1, 2,$$

and the functions $\{\varphi_{k,n}: n = 1, 2, \ldots\}$ *are as in Theorem* 2.A. *In particular*,

$$\mathbf{E}|I_1(t_1, t_2)|^2 = \mathbf{E}\left[\int_{t_1}^{t_2} |f_1(\tau, t_2)|^2\, d\tau + \left|\int_{t_1}^{t_2} f_1(\tau, t_2)\, d\tau\right|^2\right] + 2\mathbf{E}\int_{t_1}^{t_2} f_1(\tau, t_2) g_1(\tau, t_2)\, d\tau$$
$$= \mathbf{E}\int_{t_1}^{t_2} |f_1(\tau, t_2)|^2\, d\tau + 2\mathbf{E}\int_{t_1}^{t_2} f_1(\tau, t_2) I_1(\tau, t_2)\, d\tau, \quad 0 < t_1 < t_2.$$

PROOF. All four parts follow directly from Theorem 2.A, using (2.6), (2.7), (2.8), (2.9) and the observation

$$g_k(t_1, t_2) = I_k(t_1, t_2) - \int_{t_1}^{t_2} f_k(\tau, t_2)\, d\tau, \quad 0 < t_1 < t_2,\ k = 1, 2. \quad \square$$

3. Correction Formula. In this section we present the following

THEOREM 3.A (CORRECTION FORMULA). *Let f be $L_+^{2,1}$-adapted, and assume*

$$\int_{t_1}^{t_2} |f(t, t)|^2\, dt < \infty, \qquad \mathbf{E}\int_{t_1}^{t_2}\int_{t_1}^{t} |f(\tau, t)|^2\, d\tau\, dt < \infty.$$

Then

$$\int_{t_1}^{t_2}\int_{\tau}^{t_2} f(\tau, t)\, d\beta(t)\, d\beta(\tau) = \int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau, t)\, d\beta(\tau)\, d\beta(t) + \int_{t_1}^{t_2} f(t, t)\, dt.$$

PROOF. If f is a deterministic function, the result follows directly from (2.6) with $n = 1$ and $\varphi(t_1, t_2; \tau_1) = f(t_1, \tau_1)$. So let f be of the monomial form

$$f(\tau, t) = \int_{T_n(\tau, t)} \varphi(\tau, t; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \qquad t_1 < \tau < t < t_2, \tag{3.1}$$

where $n \geq 1$. Then by (2.6)

$$\int_{t_1}^{t} f(\tau, t)\, d\beta(\tau) = \int_{T_{n+1}(t_1,t)} \varphi(\tau_1, t; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1}) + \int_{t_1}^{t} f^{\wedge}(\tau, t)\, d\tau, \qquad t_1 \leq t \leq t_2,$$

and thus

$$\int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau, t)\, d\beta(\tau)\, d\beta(t) = \int_{T_{n+2}(t_1,t_2)} \varphi(\tau_1, \tau_{n+2}; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+2}) + \int_{t_1}^{t_2}\int_{\tau}^{t_2} f^{\wedge}(\tau, t)\, d\beta(t)\, d\tau,$$

where we have used a Fubini theorem in manipulating the last term. Next let

$$g(\tau, t_2) = \int_{\tau}^{t_2} f(\tau, t)\, d\beta(t), \qquad t_1 \leq \tau \leq t_2,$$

so that

$$g(\tau, t_2) = \int_{T_{n+1}(\tau,t_2)} \varphi(\tau, \tau_{n+1}; \tau_1, \ldots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1}), \qquad t_1 \leq \tau \leq t_2.$$

Then by (2.6)

$$\int_{t_1}^{t_2}\int_{\tau}^{t_2} f(\tau, t)\, d\beta(t)\, d\beta(\tau) = \int_{t_1}^{t_2} g(\tau, t_2)\, d\beta(\tau)$$

$$= \int_{T_{n+2}(t_1,t_2)} \varphi(\tau_1, \tau_{n+2}; \tau_2, \ldots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+2}) + \int_{t_1}^{t_2} g^{\wedge}(\tau, t_2)\, d\tau.$$

Since $f(t, t) \equiv 0$ it is enough to show that

$$g^{\wedge}(\tau, t_2) = \int_{\tau}^{t_2} f^{\wedge}(\tau, t)\, d\beta(t), \qquad t_1 \leq \tau \leq t_2,$$

and this is easily verified from the forms of f and g. Thus the Correction Formula holds if f is of the form (3.1). Finally, using Theorem 2.A and the continuity condition of Theorem 2.B, it follows that the Correction Formula holds for any $L_+^{2,1}$-adapted process f. □

The process

$$\eta(t_1, t_2) = \int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)$$

is, by Theorem 2.C, $L_+^{2,1}$-adapted, and, as such, possesses a formal differential

$\partial_{t_1}\eta(t_1, t_2)$. However, it is of greater interest to compute $\partial_{t_2}(t_1, t_2)$ since this is an Itô-differential (not just a formal notation), and

$$\eta(t_1, t_2) = \int_{t_1}^{t_2} \partial_\tau \eta(t_1, \tau).$$

The Correction Formula can be employed to this end.

THEOREM 3.B. *Let f be $L_+^{2,1}$-adapted and set*

$$\eta(t_1, t_2) = \int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau), \qquad 0 < t_1 < t_2.$$

Suppose

$$f(t_1, t_2) - f(t_1, t_1) = \int_{t_1}^{t_2} a(t_1, \tau)\, d\tau + \int_{t_1}^{t_2} b(t_1, \tau)\, d\beta(\tau), \qquad 0 < t_1 < t_2,$$

where a, b are $L_+^{2,1}$-adapted processes satisfying

$$\mathbf{E}\int_0^T \int_0^{t_2} \left[|a(t_1, t_2)|^2 + |b(t_1, t_2)|^2 \right] dt_1\, dt_2 < \infty, \qquad T > 0,$$

$$\int_0^T |b(t, t)|^2\, dt < \infty, \qquad T > 0.$$

Then

$$\partial_{t_2}\eta(t_1, t_2) = \left[b(t_2, t_2) + \int_{t_1}^{t_2} a(\tau, t_2)\, d\beta(\tau) \right] dt_2 + \left[f(t_2, t_2) + \int_{t_1}^{t_2} b(\tau, t_2)\, d\beta(\tau) \right] d\beta(t_2).$$

PROOF. The theorem follows directly from the Correction Formula. Indeed,

$$\int_{t_1}^{t_2} \left[b(\tau, \tau) + \int_{t_1}^{\tau} a(\tau', \tau)\, d\beta(\tau') \right] d\tau + \int_{t_1}^{t_2} \left[f(\tau, \tau) + \int_{t_1}^{\tau} b(\tau', \tau)\, d\beta(\tau') \right] d\beta(\tau)$$

$$= \int_{t_1}^{t_2} f(\tau', \tau')\, d\beta(\tau') + \int_{t_1}^{t_2} \left[\int_{\tau'}^{t_2} a(\tau', \tau)\, d\tau + \int_{\tau'}^{t_2} b(\tau', \tau)\, d\beta(\tau) \right] d\beta(\tau')$$

$$= \int_{t_1}^{t_2} f(\tau', t_2)\, d\beta(\tau'). \quad \square$$

It is worthy of note that although the process (here t_1 is fixed) $x(t) = \int_{t_1}^{t} f(\tau, t)\, d\beta(\tau)$ is not, in general, a martingale, the Correction Formula does provide its Doob-Meyer decomposition whenever f is of the form (cf. (2.2))

$$f(t_1, t_2) = \int_{t_1}^{t_2} \psi(t_1, \tau)\, d\beta(\tau).$$

Thus

$$\int_{t_1}^{t_2} f(\tau, t)\, d\beta(\tau) = \int_{t_1}^{t}\left[\int_{t_1}^{\tau} \psi(\tau_1, \tau)\, d\beta(\tau_1)\right] d\beta(\tau) + \int_{t_1}^{t} \psi(\tau, \tau)\, d\tau. \quad (3.2)$$

As another application of the Correction Formula, let $\lambda(t)$ be a strictly increasing differentiable function of t on $[t_1, t_2]$ with

$$\lambda(t) > t, \qquad t_1 \leqslant t \leqslant t_2. \quad (3.3)$$

Suppose we were to define, for $t_1 \leqslant \tau \leqslant t \leqslant t_2$,

$$f(\tau, t) = \begin{cases} 1, & t \leqslant \lambda(\tau), \\ 0, & t > \lambda(\tau), \end{cases}$$

and substitute this in the Correction Formula. Then

$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))\, d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))\, d\beta(\tau)$$
$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)). \quad (3.4)$$

This is an integration by parts formula. Of course the difficulty here is that f is not $L_+^{2,1}$-adapted, since $\partial_\tau f(\tau, t)$ does not exist. But in this case (since f is deterministic) the Correction Formula can be verified directly from (2.6). In fact, as long as the process

$$g(\tau, t_2) = \int_{\tau}^{t_2} f(\tau, t)\, d\beta(\tau), \qquad t_1 \leqslant \tau \leqslant t_2,$$

has a diffusional part $g\hat{}$, then

$$\int_{t_1}^{t_2} g(\tau, t_2)\, d\beta(\tau) = \int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau, t)\, d\beta(\tau)\, d\beta(t) + \int_{t_1}^{t_2} g\hat{}(\tau, t_2)\, d\tau.$$

Now we check that

$$g(\tau, t_2) = \begin{cases} \beta(\lambda(\tau)) - \beta(\tau), & t_1 \leqslant \tau \leqslant \lambda^{-1}(t_2), \\ \beta(t_2) - \beta(\tau), & \lambda^{-1}(t_2) \leqslant \tau \leqslant t_2. \end{cases}$$

Because of (3.3) it follows that $g\hat{} \equiv 1$. Thus (3.4) is established. However, a more difficult question involves the case where

$$\lambda(t) \geqslant t, \qquad t_1 \leqslant t \leqslant t_2, \quad (3.5)$$

and the strict inequality (3.3) no longer holds. Here we have

$$g^{\wedge}(\tau, t_2) = \begin{cases} 1, & \lambda(\tau) \neq \tau, \\ 1 - (1 \wedge \lambda'(\tau)), & \lambda(\tau) = \tau. \end{cases}$$

Thus we arrive at the following extension of (3.4)

$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))\, d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))\, d\beta(\tau)$$
$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)) - \int_A (1 \wedge \lambda'(\tau))\, d\tau$$

where A is the set $\{\tau \in [t_1, t_2]: \lambda(\tau) = \tau\}$. Now we merely note that, by (3.5),

$$\int_A (1 \wedge \lambda'(\tau))\, d\tau = \int_A d\tau$$

and we arrive at the following:

THEOREM 3.C (INTEGRATION BY PARTS). *Let $\lambda(t)$ be a strictly increasing differentiable function of t on $[t_1, t_2]$, with $\lambda(t) \geq t$, $t_1 \leq t \leq t_2$. Let A be the set $\{t \in [t_1, t_2]: \lambda(t) = t\}$. Then*

$$\int_{t_1}^{\lambda^{-1}(t_2)} \beta(\lambda(\tau))\, d\beta(\tau) + \int_{\lambda(t_1)}^{t_2} \beta(\lambda^{-1}(\tau))\, d\beta(\tau)$$
$$= \beta(t_2)\beta(\lambda^{-1}(t_2)) - \beta(t_1)\beta(\lambda(t_1)) - \mathfrak{L}(A)$$

where $\mathfrak{L}$ is Lebesgue measure.

Similar techniques to those used to establish Theorem 3.C can be used to generalize the Correction Formula to deterministic functions f defined on

$$S = \{(\tau, t): \lambda_1(\tau) \wedge t_2 \leq t \leq \lambda_2(\tau) \wedge t_2\}$$

where λ_1, λ_2 satisfy the conditions of Theorem 3.C, and $\lambda_1 \leq \lambda_2$. We merely extend the function f defined on S to the whole triangle, $t_1 \leq \tau \leq t \leq t_2$, by setting it equal to zero on the complement of S. The reader can check that

$$\int_S f(\tau, t)\, d\beta(t)\, d\beta(\tau) = \int_S f(\tau, t)\, d\beta(\tau)\, d\beta(t) + \int_A f(t, t)\, dt$$

where $A = S \cap \{(\tau, t): \tau = t\}$.

4. Carathéodory principle. A particularly interesting class of stochastic processes consists of those of the form

$$f(t_1, t_2) = \varphi(t_1, t_2, \beta(t_2) - \beta(t_1)), \qquad 0 \leq t_1 \leq t_2.$$

The conditions for f to be L_+^2-adapted are

$$\int_{-\infty}^{\infty} |\varphi(t_1, t_2, x)|^2 \exp(-x^2/2(t_2 - t_1))\, dx < \infty, \qquad 0 \leq t_1 \leq t_2, \quad (4.1)$$

$$\int_{-\infty}^{\infty}\int_0^{t_2}\frac{|\varphi(\tau,t_2,x)|^2}{\sqrt{t_2-\tau}}\exp(-x^2/2(t_2-\tau))\,d\tau\,dx<\infty,\qquad t_2>0. \quad (4.2)$$

And the conditions for f to be $L_+^{2,1}$-adapted are that the functions

$$\frac{\partial}{\partial x}\varphi(t_1,t_2,x),\qquad \left(\frac{\partial}{\partial t_1}-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(t_1,t_2,x)$$

also satisfy (4.1) and (4.2). For such processes f the integral $\int_{t_1}^{t_2}f(\tau,t_2)\,d\beta(\tau)$ can be related to an Itô stochastic integral. In fact we have the following result:

THEOREM 4.A (CARATHÉODORY PRINCIPLE). *Let f be an $L_+^{2,1}$-adapted process of the form*

$$f(t_1,t_2)=\varphi(t_1,t_2,\beta(t_2)-\beta(t_1)),\qquad 0\leqslant t_1\leqslant t_2.$$

Then

$$\int_{t_1}^{t_2}f(\tau,t_2)\,d\beta(\tau)=F(t_1,t_2,\beta(t_2)),\qquad 0\leqslant t_1\leqslant t_2,$$

where

$$F(t_1,t_2,x)=\int_{t_1}^{t_2}\varphi(\tau,t_2,x-\beta(\tau))\,d\beta(\tau),\qquad 0\leqslant t_1\leqslant t_2,\ x\in\mathbf{R}.$$

The proof relies on the following two lemmas.

LEMMA I. *Let H_n be the Hermite polynomial*[6] *of degree n, where $n\geqslant 1$. And let $a(t_1,t_2)$ be a deterministic function which is differentiable in t_1 and satisfies*

$$\int_0^{t_2}\left[|a(\tau,t_2)|^2+\left|\frac{\partial}{\partial\tau}a(\tau,t_2)\right|^2\right]d\tau<\infty,\qquad t_2>0.$$

Then

$$\begin{aligned}\int_{t_1}^{t_2}&a(\tau,t_2)H_n(t_2-\tau,\beta(t_2)-\beta(\tau))\,d\beta(\tau)\\ &=\frac{1}{n+1}a(t_1,t_2)H_{n+1}(t_2-t_1,\beta(t_2)-\beta(t_1))\\ &\quad+\frac{1}{n+1}\int_{t_1}^{t_2}\frac{\partial}{\partial\tau}a(\tau,t_2)H_{n+1}(t_2-\tau,\beta(t_2)-\beta(\tau))\,d\tau\\ &\quad+n\int_{t_1}^{t_2}a(\tau,t_2)H_{n-1}(t_2-\tau,\beta(t_2)-\beta(\tau))\,d\tau,\qquad 0\leqslant t_1\leqslant t_2.\end{aligned}$$

[6]These polynomials are defined by

$$H_n(t,x)=(-t)^n\exp(x^2/2t)\frac{\partial^n}{\partial x^n}\exp(-x^2/2t),\qquad t>0,\ x\in\mathbf{R},\ n=0,1,\ldots.$$

LEMMA II. *Theorem* 4.A *holds for functions* φ *of the form*

$$\varphi(t_1, t_2, x) = a(t_1, t_2)H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R},$$

where a satisfies the conditions of Lemma I.

PROOF OF LEMMA I. The proof relies on the fact that

$$\frac{1}{n!} H_n(t_2 - t_1, \beta(t_2) - \beta(t_1)) = \int_{T_n(t_1,t_2)} d\beta(\tau_1) \cdots d\beta(\tau_n), \qquad n = 1, 2, \ldots.$$

(A very short proof of this result appears in McKean [**12**, p. 37].) Thus, using (2.6) and substituting

$$a(\tau, t_2) = a(t_1, t_2) + \int_{t_1}^{\tau} \frac{\partial}{\partial \tau_1} a(\tau_1, t_2)\, d\tau_1, \qquad t_1 < \tau < t_2,$$

we have

$$\begin{aligned}
\frac{1}{n!} \int_{t_1}^{t_2} & a(\tau, t_2) H_n(t_2 - \tau, \beta(t_2) - \beta(\tau))\, d\beta(\tau) \\
&= \int_{T_{n+1}(t_1,t_2)} a(\tau_1, t_2)\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1}) \\
&\quad + \int_{t_1}^{t_2} a(\tau, t_2) \int_{T_{n-1}(\tau,t_2)} d\beta(\tau_1) \cdots d\beta(\tau_{n-1})\, d\tau \\
&= a(t_1, t_2) \int_{T_{n+1}(t_1,t_2)} d\beta(\tau_1) \cdots d\beta(\tau_{n+1}) \\
&\quad + \int_{t_1}^{t_2} \frac{\partial}{\partial \tau} a(\tau, t_2) \int_{T_{n+1}(\tau,t_2)} d\beta(\tau_1) \cdots d\beta(\tau_{n+1})\, d\tau \\
&\quad + \int_{t_1}^{t_2} a(\tau, t_2) \int_{T_{n-1}(\tau,t_2)} d\beta(\tau_1) \cdots d\beta(\tau_{n-1})\, d\tau \\
&= \frac{1}{(n+1)!} a(t_1, t_2) H_{n+1}(t_2 - t_1, \beta(t_2) - \beta(t_1)) \\
&\quad + \frac{1}{(n+1)!} \int_{t_1}^{t_2} \frac{\partial}{\partial \tau} a(\tau, t_2) H_{n+1}(t_2 - \tau, \beta(t_2) - \beta(\tau))\, d\tau \\
&\quad + \frac{1}{(n-1)!} \int_{t_1}^{t_2} a(\tau, t_2) H_{n-1}(t_2 - \tau, \beta(t_2) - \beta(\tau))\, d\tau,
\end{aligned}$$

from which the desired result follows. □

PROOF OF LEMMA II. Let

$$\varphi_\lambda(t_1, t_2, x) = a(t_1, t_2)\exp(\lambda x - \lambda^2(t_2 - t_1)/2), \qquad 0 < t_1 < t_2; \lambda, x \in \mathbf{R}.$$

We now evaluate the integral below using Itô's Formula.

$$\begin{aligned} F_\lambda(t_1, t_2, x) &= \int_{t_1}^{t_2} \varphi_\lambda(\tau, t_2, x - \beta(\tau))\, d\beta(\tau) \\ &= \frac{1}{\lambda}\varphi_\lambda(t_1, t_2, x - \beta(t_1)) - \frac{1}{\lambda} a(t_2, t_2)\exp\big(\lambda[x - \beta(t_2)]\big) \\ &\quad + \lambda \int_{t_1}^{t_2} \varphi_\lambda(\tau, t_2, x - \beta(\tau))\, d\tau \\ &\quad + \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial \tau} a(\tau, t_2)\exp\big(\lambda[x - \beta(\tau)] - \lambda^2(t_2 - \tau)/2\big)\, d\tau, \\ &\qquad\qquad 0 < t_1 < t_2, x \in \mathbf{R}, \end{aligned}$$

and thus

$$\begin{aligned} F_\lambda(t_1, t_2, \beta(t_2)) &= \frac{1}{\lambda}\varphi_\lambda(t_1, t_2, \beta(t_2) - \beta(t_1)) - \frac{1}{\lambda} a(t_1, t_2) \\ &\quad + \lambda \int_{t_1}^{t_2} \varphi_\lambda(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\tau \\ &\quad + \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial \tau} a(\tau, t_2)\Big\{\exp\big(\lambda[\beta(t_2) - \beta(\tau)] - \tfrac{1}{2}\lambda^2(t_2 - \tau)\big) - 1\Big\}\, d\tau, \\ &\qquad\qquad 0 < t_1 < t_2. \end{aligned}$$

On the other hand,

$$\varphi_\lambda(t_1, t_2, x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a(t_1, t_2) H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R},$$

and thus, by Lemma I,

$$\begin{aligned} &\int_{t_1}^{t_2} \varphi_\lambda(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\beta(\tau) \\ &\quad = \frac{1}{\lambda}\varphi_\lambda(t_1, t_2, \beta(t_2) - \beta(t_1)) - \frac{1}{\lambda} a(t_1, t_2) \\ &\qquad + \frac{1}{\lambda}\int_{t_1}^{t_2} \frac{\partial}{\partial \tau} a(\tau, t_2)\Big\{\exp\big(\lambda[\beta(t_2) - \beta(\tau)] - \tfrac{1}{2}\lambda^2(t_2 - \tau)\big) - 1\Big\}\, d\tau \\ &\qquad + \lambda \int_{t_1}^{t_2} \varphi_\lambda(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\tau, \qquad 0 < t_1 < t_2. \end{aligned}$$

And from this it follows that Theorem 4.A holds for $\{\varphi_\lambda : \lambda \in \mathbf{R}\}$; that is,

$$\int_{t_1}^{t_2} {}_{\lambda}(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\beta(\tau) = F_\lambda(t_1, t_2, \beta(t_2)),$$

$$0 < t_1 < t_2, \lambda \in \mathbf{R}.$$

By differentiating this equation n times with respect to λ, and setting $\lambda = 0$, it follows that Theorem 4.A holds for the function

$$\varphi(t_1, t_2, x) = a(t_1, t_2)H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R}. \quad \square$$

Now we are in a position to present the

PROOF OF THEOREM 4.A. Let φ satisfy (4.1). Then, because of the completeness of the Hermite polynomials for any fixed first argument, there exists a unique sequence $\{a_n(t_1, t_2): 0 < t_1 < t_2, n = 0, 1, \ldots\}$ such that

$$\varphi(t_1, t_2, x) = \sum_{n=0}^{\infty} a_n(t_1, t_2)H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R}, \quad (4.3)$$

in the sense that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left|\varphi(t_1, t_2, x) - \sum_{n=0}^{N} a_n(t_1, t_2)H_n(t_2 - t_1, x)\right|^2 \cdot \exp\big(-x^2/2(t_2 - t_1)\big)\, dx = 0, \qquad 0 < t_1 < t_2.$$

Furthermore, since

$$\frac{\partial}{\partial x} H_n(t, x) = nH_{n-1}(t, x), \qquad t > 0, x \in \mathbf{R}, n = 1, 2, \ldots,$$

it follows that if $\partial\varphi(t_1, t_2, x)/\partial x$ satisfies (4.1) then

$$\frac{\partial}{\partial x}\varphi(t_1, t_2, x) = \sum_{n=0}^{\infty} a_n(t_1, t_2)\frac{\partial}{\partial x}H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R},$$

in the same sense. Finally, since

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)H_n(t, x) \equiv 0, \qquad t > 0, x \in \mathbf{R}, n = 0, 1, \ldots,$$

it likewise follows that if $(\partial/\partial t_1 - \frac{1}{2}(\partial^2/\partial x^2))\varphi(t_1, t_2, x)$ satisfies (4.1) then

$$\left(\frac{\partial}{\partial t_1} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(t_1, t_2, x) = \sum_{n=0}^{\infty}\left[\frac{\partial}{\partial t_1}a_n(t_1, t_2)\right]H_n(t_2 - t_1, x), \qquad 0 < t_1 < t_2, x \in \mathbf{R},$$

in the same sense.

Now we define for $n = 0, 1, \ldots$

$$f_n(t_1, t_2) = a_n(t_1, t_2)H_n(t_2 - t_1, \beta(t_2) - \beta(t_1)), \qquad 0 < t_1 < t_2.$$

Because of (4.3) and the continuity condition of Theorem 2.B it follows that, for $0 < t_1 < t_2$,

$$\sum_{n=0}^{N} \int_{t_1}^{t_2} f_n(\tau, t_2)\, d\beta(\tau)$$

tends to $\int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)$ in $L^2(\Omega)$ for large N. Furthermore, if

$$F_n(t_1, t_2, x) = \int_{t_1}^{t_2} a_n(\tau, t_2) H_n(t_2 - \tau, x - \beta(\tau))\, d\beta(\tau),$$

$$0 < t_1 < t_2, n = 0, 1, \ldots,$$

then $\sum_{n=0}^{N} F_n(t_1, t_2, x)$ tends to $F(t_1, t_2, x)$ in $L^2(\Omega)$ in the sense that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \mathbf{E}\left| F(t_1, t_2, x) - \sum_{n=0}^{N} F_n(t_1, t_2, x) \right|^2$$

$$\cdot \exp(-x^2/2(t_2 - t_1))\, dx = 0, \qquad 0 < t_1 < t_2,$$

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \mathbf{E}\left| \frac{\partial}{\partial x} F(t_1, t_2, x) - \sum_{n=0}^{N} \frac{\partial}{\partial x} F_n(t_1, t_2, x) \right|^2$$

$$\cdot \exp(-x^2/2(t_2 - t_1))\, dx = 0, \qquad 0 < t_1 < t_2.$$

Since this implies that $\sum_{n=0}^{N} F_n(t_1, t_2, \beta(t_2))$ tends to $F(t_1, t_2, \beta(t_2))$ in $L^2(\Omega)$ for large N, $0 < t_1 < t_2$, and since, by Lemma II, for $n = 1, 2, \ldots$

$$\int_{t_1}^{t_2} f_n(\tau, t_2)\, d\beta(\tau) = F_n(t_1, t_2, \beta(t_2)), \qquad 0 < t_1 < t_2,$$

the proof of Theorem 4.A is complete. □

As a corollary of Theorem 3.B we present the following result.

THEOREM 4.B. *Let*

$$\eta(t_1, t_2) = \int_{t_1}^{t_2} \varphi(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\beta(\tau),$$

where

$$\varphi(t_1, t_2, x), \quad \frac{\partial}{\partial x}\varphi(t_1, t_2, x), \quad \left(\frac{\partial}{\partial t_1} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(t_1, t_2, x),$$

$$\left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(t_1, t_2, x)$$

satisfy (4.1) *and* (4.2). *Then*

$$\partial_{t_2}\eta(t_1, t_2) = \left[\varphi(t_2, t_2, 0) + \int_{t_1}^{t_2} \frac{\partial}{\partial x}\varphi(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\beta(\tau)\right] d\beta(t_2)$$

$$+ \left[\frac{\partial}{\partial x}\varphi(t_2, t_2, 0) + \int_{t_1}^{t_2}\left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(\tau, t_2, \beta(t_2) - \beta(\tau))\, d\beta(\tau)\right] dt_2,$$

$$0 < t_1 < t_2.$$

PROOF. The result follows directly from Theorem 3.B once we observe that, by Itô's Formula,

$$\partial_{t_2}\varphi(t_1, t_2, \beta(t_2) - \beta(t_1)) = \frac{\partial}{\partial x}\varphi(t_1, t_2, \beta(t_2) - \beta(t_1))\, d\beta(t_2)$$
$$+ \left(\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\varphi(t_1, t_2, \beta(t_2) - \beta(t_1))\, dt_2. \quad \square$$

5. Stochastic integral equation. In this section we study the linear stochastic integral equation

$$\xi(t) - \int_0^t \sigma(\tau, t)\xi(\tau)\, d\beta(\tau) - \int_0^t b(\tau, t)\xi(\tau)\, d\tau = F(t), \qquad t \geqslant 0, \quad \text{(SIE)}$$

where σ, b, F are deterministic functions. A more general class of equations is analyzed in Berger [2] and Berger and Mizel [3]; but to make this exposition self-contained, the existence-uniqueness result for (SIE) are presented here.

THEOREM 5.A. *Let σ, b, F be deterministic functions satisfying*

$$\|\sigma\|_T \equiv \sup_{0 \leqslant t_1 \leqslant t_2 \leqslant T} |\sigma(t_1, t_2)| < \infty,$$
$$\|b\|_T \equiv \sup_{0 \leqslant t_1 \leqslant t_2 \leqslant T} |b(t_1, t_2)| < \infty,$$
$$\|F\|_T \equiv \sup_{0 \leqslant t \leqslant T} |F(t)| < \infty,$$

for each $T \geqslant 0$. Then there exists a solution $\xi(t)$ of (SIE) *on* $[0, T]$ *for any $T \geqslant 0$ such that*

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}|\xi(t)|^2 < \infty. \tag{5.1}$$

Furthermore, if $\tilde{\xi}$ is another solution of (SIE) *satisfying* (5.1), *then $\tilde{\xi}$ is a version of ξ.*

PROOF. To establish existence we construct the successive approximants to (SIE). Thus let

$$\xi_0(t) = F(t), \qquad t \geqslant 0$$

$$\xi_n(t) = F(t) + \int_0^t \sigma(\tau, t)\xi_{n-1}(\tau)\, d\beta(\tau) + \int_0^t b(\tau, t)\xi_{n-1}(\tau)\, d\tau,$$
$$t \geqslant 0, n = 1, 2, \ldots. \tag{5.2}$$

The first property of these iterates we establish is

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}|\xi_n(t)|^2 < \infty, \qquad T \geqslant 0, n = 1, 2, \ldots.$$

This is shown by induction as follows.

$$\sup_{0 \leq t \leq T} \mathbf{E}|\xi_n(t)|^2 \leq 3\|F\|_T^2 + 3N_T \sup_{0 \leq t \leq T} \mathbf{E}|\xi_{n-1}(t)|^2, \qquad T \geq 0, n = 1, 2, \ldots,$$

where

$$N_T = \sup_{0 \leq t \leq T} \left[\int_0^t |\sigma(\tau, t)|^2 \, d\tau + t \int_0^t |b(\tau, t)|^2 \, d\tau \right], \qquad T \geq 0.$$

The next property we establish is

$$\sup_{0 \leq t \leq T} \mathbf{E}|\xi_{n+1}(t) - \xi_n(t)|^2 \leq 2N_T\left(1 + \|F\|_T^2\right) \frac{(2M_T)^n}{n!}, \qquad T \geq 0, \quad (5.3)$$

where

$$M_T = \|\sigma\|_T^2 + T\|b\|_T^2, \qquad T \geq 0.$$

This is shown by the following observation,

$$\mathbf{E}|\xi_{n+1}(t) - \xi_n(t)|^2 \leq 2M_T \int_0^T \mathbf{E}|\xi_n(\tau) - \xi_{n-1}(\tau)|^2 \, d\tau,$$
$$0 \leq t \leq T, n = 1, 2, \ldots.$$

Thus, by (5.3), for each $t \in [0, T]$, the sequence $\{\xi_n(t)\}$ converges in $L^2(\Omega)$ to a random variable $\xi(t)$. The process $\xi(t)$ is $\mathcal{F}(0, t)$-measurable and

$$\sup_{0 \leq t \leq T} \mathbf{E}|\xi(t)|^2 < \infty, \qquad T \geq 0.$$

Since

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \mathbf{E}|\xi_n(t) - \xi(t)|^2 = 0, \qquad T \geq 0,$$

taking limits in (5.2) is valid and $\xi(t)$ is, therefore, a solution of (SIE).

To establish uniqueness let $\xi(t)$ and $\tilde{\xi}(t)$ denote two solutions of (SIE) satisfying (5.1). Then

$$\mathbf{E}|\xi(t) - \tilde{\xi}(t)|^2 \leq 2M_T \int_0^t \mathbf{E}|\xi(\tau) - \tilde{\xi}(\tau)|^2 \, d\tau, \qquad 0 \leq t \leq T,$$

and thus

$$\mathbf{E}|\xi(t) - \tilde{\xi}(t)|^2 = 0, \qquad 0 \leq t \leq T. \quad \square$$

The successive approximants (5.2) are particularly interesting in view of the Correction Formula. In fact the solution of (SIE) can be represented as an adapted stochastic integral. This is the content of the following

Theorem 5.B (Resolvent Formula). *Let σ, b, F be as in Theorem* 5.A, *and also satisfy*

$$\sup_{0<t_1<t_2<T}\left[\left|\frac{\partial}{\partial t_1}\sigma(t_1,t_2)\right|+\left|\frac{\partial}{\partial t_1}b(t_1,t_2)\right|\right]<\infty, \qquad T>0, \tag{5.4}$$

$$\sup_{0<t<T}\left|\frac{d}{dt}F(t)\right|<\infty, \qquad T>0. \tag{5.5}$$

Define the iterates σ_n, b_n as follows:

$$\sigma_1(t_1,t_2)=\sigma(t_1,t_2), \quad b_1(t_1,t_2)=b(t_1,t_2), \qquad 0<t_1<t_2,$$

$$\sigma_{n+1}(t_1,t_2)=\int_{t_1}^{t_2}\sigma_n(t_1,\tau)\sigma(\tau,t_2)\,d\beta(\tau)+\int_{t_1}^{t_2}\sigma_n(t_1,\tau)b(\tau,t_2)\,d\tau,$$
$$0<t_1<t_2, n=1,2,\ldots,$$

$$b_{n+1}(t_1,t_2)=\int_{t_1}^{t_2}b_n(t_1,\tau)\sigma(\tau,t_2)\,d\beta(\tau)+\int_{t_1}^{t_2}b_n(t_1,\tau)b(\tau,t_2)\,d\tau,$$
$$0<t_1<t_2, n=1,2,\ldots.$$

Then the resolvents

$$r_\sigma(t_1,t_2)=\sum_{n=1}^{\infty}\sigma_n(t_1,t_2), \quad r_b(t_1,t_2)=\sum_{n=1}^{\infty}b_n(t_1,t_2), \quad 0<t_1<t_2,$$

exist and are $L_+^{2,1}$-adapted processes. Furthermore the solution to (SIE) *is*

$$\xi(t)=F(t)+\int_0^t r_\sigma(\tau,t)F(\tau)\,d\beta(\tau)$$
$$+\int_0^t\left[r_b(\tau,t)-\sigma(\tau,\tau)r_\sigma(\tau,t)\right]F(\tau)\,d\tau, \qquad t>0.$$

Proof. This result is actually a corollary of Theorem 5.A. Indeed, by the Correction Formula, it follows that the successive approximants ξ_n are given by

$$\xi_n(t)=F(t)+\int_0^t\left[\sum_{k=1}^{n}\sigma_k(\tau,t)\right]F(\tau)\,d\beta(\tau)$$
$$+\int_0^t\left[\sum_{k=1}^{n}b_k(\tau,t)-\sigma(\tau,\tau)\sum_{k=1}^{n-1}\sigma_k(\tau,t)\right]F(\tau)\,d\tau,$$
$$t>0, n=2,3,\ldots. \tag{5.6}$$

Thus the convergence of the successive approximants implies the existence of r_σ, r_b. The conditions (5.4) and (5.5), together with the continuity condition of Theorem 2.B, allow us to take limits in (5.6). □

Actually, because of the restrictive L^∞ assumptions on σ and b, the convergence of the approximants ξ_n is almost sure convergence. This is because there exists a function $C(t)$ such that

$$\mathbf{E}|\xi_{n+1}(t) - \xi_n(t)|^2 < \frac{C^n(t)}{n!}, \qquad t > 0.$$

This is actually the content of (5.3). And thus the series

$$\sum_{n=1}^{\infty} \mathbf{P}\left\{ |\xi_{n+1}(t) - \xi_n(t)| > \frac{1}{n^2} \right\}$$

converges for each $t \geqslant 0$. So that by the Borel-Cantelli Lemma, $\xi_n(t)$ converges almost surely for each $t \geqslant 0$. Similarly the conditions (5.4), (5.5) imply the almost sure convergence of the terms in (5.6). And thus the Resolvent Formula provides trajectory-type information. For examples concerning the use of the Resolvent Formula, and for additional information about the solution of (SIE), and for the case where σ, b, F are processes themselves, the reader is referred to Berger [2], and Berger and Mizel [3].

6. Related stochastic integrals. The authors gratefully acknowledge the help of the referee in describing the adapted stochastic integral of §2 in the framework of recent work of Itô [9] and Ogawa [15]. If f is $L_+^{2,1}$-adapted then

$$I^+(t_1, t_2; f) \equiv \int_{t_1}^{t_2} f(\tau, t_2)\, d\beta(\tau)$$

can be written as

$$\int_{t_1}^{t_2} f^*(t_1, \tau)\, d\beta^*(\tau) \tag{6.1}$$

where

$$f^*(t_1, \tau) = f(t_1 + t_2 - \tau, t_2), \qquad \beta^*(\tau) = \beta(t_1 + t_2) - \beta(t_1 + t_2 - \tau).$$

Furthermore, f^* is L_-^2-adapted with respect to β^*. However, the integral (6.1) does not correspond to Itô's classical integral,

$$I^-(t_1, t_2; f^*) \equiv \int_{t_1}^{t_2} f^*(t_1, \tau)\, d\beta^*(\tau),$$

as defined in Itô [7]. The reason for this is as follows. It is shown in Berger [2] that $I^+(t_1, t_2; f)$ can be written as

$$\lim_{\delta \downarrow 0} \sum_{i=1}^{n} f(s_i, t_2)\left[\beta(s_{i+1}) - \beta(s_i)\right]$$

where $t_1 = s_1 < \cdots < s_{n+1} = t_2$ is a partition of $[t_1, t_2]$, and

$$\delta = \max_{i=1,\ldots,n} (s_{i+1} - s_i).$$

And this limit is also

$$\lim_{\delta\downarrow 0} \sum_{i=1}^{n} f^*(t_1, s_{i+1}^*)[\beta^*(s_{i+1}^*) - \beta^*(s_i^*)],$$

where $s_{i+1}^* = t_1 + t_2 - s_{n+1-i}^*$. And this corresponds to what is referred to as an I_1-integral, $I_1^-(t_1, t_2; f^*)$. In fact, (2.5) amounts to

$$I_1^-(t_1, t_2; f^*) = I^-(t_1, t_2; f^*) + \int_{t_1}^{t_2} f'(\tau, t_2)\, d\tau. \tag{6.2}$$

It is important to note here that whereas for the classical integral $I^-(t_1, t_2; f^*)$ to exist it is enough that f^* be L^2-adapted, for $I_1^-(t_1, t_2; f^*)$ to exist it is necessary in addition that f' exist. This is related to the notion of β-differentiability introduced in Ogawa [15].

The reader can check that in the context of (6.2) the Correction Formula becomes

$$\int_{t_1}^{t_2}\left[\int_{\tau}^{t_2} f(\tau, t)\, d\beta(t)\right]^* d\beta^*(\tau) = \int_{t_1}^{t_2}\left[\int_{t_1}^{t} f^*(\tau, t)\, d\beta^*(\tau)\right] d\beta(t),$$

and the Resolvent Formula becomes

$$\xi(t) = F(t) + \int_0^t [r_\sigma(\tau, t)F(\tau)]^* d\beta^*(\tau)$$

$$+ \int_0^t r_b(\tau, t)F(\tau)\, d\tau, \qquad t \geq 0.$$

All of these results can be expanded to more general I_α-integrals, defined by

$$I_\alpha^-(t_1, t_2; f^*) = \lim_{\delta\downarrow 0} \sum_{i=1}^{n} f^*(t_1, (1-\alpha)s_i + \alpha s_{i+1})[\beta^*(s_{i+1}) - \beta^*(s_i)].$$

The interested reader is referred to Berger [2] for further details.

REFERENCES

1. L. Arnold, *Stochastic differential equations: Theory and applications*, Wiley-Interscience, New York, 1974.

2. M. A. Berger, *Stochastic Ito-Volterra equations*, Ph.D. Dissertation, Carnegie-Mellon Univ., Pittsburgh, Pa., 1977.

3. M. A. Berger and V. J. Mizel, *A Fubini theorem for iterated stochastic integrals*, Bull. Amer. Math. Soc. **84** (1978), 159–160.

4. A. T. Bharucha-Reid, *Random integral equations*, Academic Press, New York, 1972.

5. A. Friedman, *Stochastic differential equations and applications*. Vol. I, Academic Press, New York, 1975.

6. I. I. Gihman and A. V. Skorohod, *Stochastic differential equations*, Springer-Verlag, New York, 1972.

7. K. Itô, *Stochastic integral*, Proc. Imp. Acad. Tokyo **20** (1944), 519–524.

8. ______, *Multiple Wiener integral*, J. Math. Soc. Japan 3 (1951), 157–169.

9. ______, *Stochastic differentials*, Appl. Math. Optim. 1 (1975), 374–381.

10. A. Kufner, O. John and S. Fucik, *Function spaces*, Noordhoff International, Leyden, 1977.

11. J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*. Vol. I, Springer-Verlag, New York, 1972

12. H. P. McKean, *Stochastic integrals*, Academic Press, New York, 1969.

13. E. J. McShane, *Stochastic calculus and stochastic models*, Academic Press, New York, 1974.

14. P. S. Meyer, *Séminaire de probabilités*. X, Lecture Notes in Math., vol. 511, Springer-Verlag, New York, 1976.

15. S. Ogawa, *On a Riemann definition of the stochastic integral*. I, Proc. Japan Acad. **46** (1970), 153–157.

16. A. V. Skorohod, *Studies in the theory of random processes*, Addison-Wesley, Reading, Mass., 1965.

17. C. P. Tsokos and W. J. Padgett, *Random integral equations with applications to life sciences and engineering*, Academic Press, New York, 1974.

Mathematical Research Center, University of Wisconsin, 610 N. Walnut St., Madison, Wisconsin 53706

Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213 (Current address of V. J. Mizel)

Current address (M. A. Berger): Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332